The Hamilton Equations of Motion

→ We consider holonomic mechanical systems where the forces are derived from a potential dependent only on positions.

In the Lagrangian formulation a system with n degrees of freedom has n equations of motion of the form:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = 0 \qquad i=1,2,\dots,n \tag{1}$$

These are n second order equations, so that the system is determined for all time only when 2n initial values are specified, e.g. n q_i's and n $\dot{q}_i$'s at a particular time t_s.

The Hamiltonian formulation is based on a different picture:
We seek to describe the motion in terms of first order equations. The number of initial conditions determining the motion must still be 2n, so there must be 2n independent first order equations expressed in terms of 2n independent variables. The formulation is nearly symmetric if we choose the 2n independent variables to be:
the n generalized coordinates q_i, and
the n conjugate momenta p_i:

$$p_i = \frac{\partial L(q,\dot{q},t)}{\partial \dot{q}_i} \tag{2}$$

The quantities (q, p) are the so-called canonical variables

From a mathematical viewpoint the transition from Lagrangian to Hamiltonian formulation corresponds to changing the variables of our mechanical functions from $(q, \dot{q}, t)$ to (q, p, t) where p is related to q and $\dot{q}$ by eqs. (2). Eqs. (2) can be in general solved in order to get:

$$\dot{q}_i = q_i(q_1, q_2, \ldots, q_n, p_1, p_2 \ldots p_n, t) \tag{3}$$

The Hamiltonian methods are not particularly superior to Lagrangian techniques for the direct solution of mechanical problems.

The usefulness of the Hamiltonian viewpoint lies in providing a framework for theoretical extensions in many areas of physics.

We define as the **Hamiltonian function**:

$$\boxed{H(q, p, t) = \sum_i \dot{q}_i p_i - L(q, \dot{q}, t)} \tag{4}$$

where H is considered a function of q, p, t, so that:

$$dH = \sum_i \left(\frac{\partial H}{\partial q_i} dq_i + \frac{\partial H}{\partial p_i} dp_i \right) + \frac{\partial H}{\partial t} dt \tag{5}$$

The r.h.s of (4) gives:

$$dH = \sum_i \left(\dot{q}_i dp_i + \cancel{p_i d\dot{q}_i} - \cancel{\frac{\partial L}{\partial \dot{q}_i} d\dot{q}_i} - \frac{\partial L}{\partial q_i} dq_i \right) - \frac{\partial L}{\partial t} dt \tag{6}$$

Since $p_i = \frac{\partial L}{\partial \dot{q}_i}$ (eq. (3)) and $\dot{p}_i = \frac{\partial L}{\partial q_i}$ (eq. (1)), eq. (6) becomes:

$$dH = \sum_i \left(\dot{q}_i dp_i - \dot{p}_i dq_i \right) - \frac{\partial L}{\partial t} dt \tag{7}$$

Since dq_i, dp_i, dt are independent their coefficients in eqs. (5) and (7) must be equal. So we get the canonical equations of Hamilton:

$$\boxed{\begin{aligned} \dot{q}_i &= \frac{\partial H}{\partial p_i} \\ \dot{p}_i &= -\frac{\partial H}{\partial q_i} \qquad i=1,2,\dots,n \\ \frac{\partial H}{\partial t} &= -\frac{\partial L}{\partial t} \end{aligned}} \qquad (8)$$

Remark: The Hamiltonian (eq. (4)) has the same value as h

$$h = \sum_i \dot{q}_i \frac{\partial L}{\partial \dot{q}_i} - L(q, \dot{q}, t) \qquad (9)$$

the energy function or Jacobi's integral But h is a function of $q, \dot{q}, t$ while H is a function of q, p, t.

The $2n$ first order equations (8) describe the behavior of the system which is represented by a point in a phase space whose coordinates are the $2n$ independent canonical variables q, p.

Construction of the Hamiltonian

Example: Planar motion in a central force potential

1) Construct the Lagrangian $L(q, \dot{q}, t)$ for a choosen set of generalized coordinates q_i

Using the polar coordinates r, ϑ as generalized coordinates we have the equations of transformation:

$$\left.\begin{aligned} x &= r\cdot\cos\vartheta \\ y &= r\cdot\sin\vartheta \end{aligned}\right\} \Longrightarrow \begin{aligned} \dot{x} &= \dot{r}\cos\vartheta - r\dot{\vartheta}\sin\vartheta \\ \dot{y} &= \dot{r}\sin\vartheta + r\dot{\vartheta}\cos\vartheta \end{aligned} \qquad (9)$$

so the kinetic energy T is:

$$T = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) \Rightarrow T = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\vartheta}^2) \qquad (10)$$

The Lagrangian is:

$$L = T - V \Rightarrow L = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\vartheta}^2) - V(r) \qquad (11)$$

2) Define conjugate momenta p_i as functions of $q_i, \dot{q}_i, t$

$$\begin{aligned} p_r &= \frac{\partial L}{\partial \dot{r}} = m\dot{r} \\ p_\vartheta &= \frac{\partial L}{\partial \dot{\vartheta}} = mr^2\dot{\vartheta} \end{aligned} \qquad (12)$$

3) Use the above equations in order to obtain $\dot{q}_i$ as functions of (p, P, t)

$$\begin{aligned} \dot{r} &= \frac{p_r}{m} \\ \dot{\vartheta} &= \frac{p_\vartheta}{mr^2} \end{aligned} \qquad (13)$$

4) Construct the Hamiltonian (eq. (4)) using the above equations in order to have H as a function of q, p, t

$$H = p_r \dot{r} + p_\vartheta \dot{\vartheta} - L \xrightarrow{(12),(13)}$$

$$H = \frac{p_r^2}{2m} + \frac{p_\vartheta^2}{2mr^2} + V(r) \qquad (14)$$

The canonical equations of motion give:

$$\dot{r} = \frac{\partial H}{\partial p_r} = \frac{p_r}{m}, \quad \dot{p}_r = -\frac{\partial H}{\partial r} = -\frac{\partial V}{\partial r} + \frac{p_\vartheta^2}{mr^3}$$

$$\dot{\vartheta} = \frac{\partial H}{\partial p_\vartheta} = \frac{p_\vartheta}{mr^2}, \quad \dot{p}_\vartheta = 0$$

$$\frac{\partial H}{\partial t} = 0$$

Conservation theorems

A) Momentum

From Hamilton's equations (eq. (8)) we have:

$$\dot{p}_i = -\frac{\partial H}{\partial q_i} \quad i=1,2,\dots,n$$

If q_j is a cyclic coordinate, i.e. it does not appear explicitly in H we have:

$$\dot{p}_j = -\frac{\partial H}{\partial q_j} = 0 \Longrightarrow p_j = \text{constant} \qquad (15)$$

→ If a generalized coordinate is cyclic the conjugate momentum is conserved.

B) Energy

The total time derivative of the Hamiltonian is:

$$\frac{dH}{dt} = \sum_i \left(\frac{\partial H}{\partial q_i} \dot{q}_i + \frac{\partial H}{\partial p_i} \dot{p}_i \right) + \frac{\partial H}{\partial t} \qquad (16)$$

From the equations of motion (eq. (8)) we see that the sum vanishes so we have:

$$\frac{dH}{dt} = \frac{\partial H}{\partial t} = -\frac{\partial L}{\partial t} \qquad (17)$$

→ If the Hamiltonian H is not an explicit function of time then H is constant in time

Modified Hamilton's principle

The Hamilton's equations of motion can be derived from a variation of Hamilton's principle:

$$\delta I \equiv \delta \int_{t_1}^{t_2} L \, dt = 0 \tag{18}$$

from which we obtain Lagrange's equations.

In the spirit of the Hamiltonian formulation, both q and p must be treated as independent coordinates so the integrand in the action integral (eq. (18)) must be expressed as a function of both q and p and their time derivatives. Another modification is that the integral must be evaluated over the trajectory of the system point in phase space, and the varied paths must be in the neighborhood of this phase space trajectory.

Modified Hamilton's principle:
The motion of a dynamical system in the phase space from time t_1 to time t_2 is such that

$$\delta I = \delta \int_{t_1}^{t_2} \Big[\sum_i p_i \dot{q}_i - H(q,p,t)\Big] dt = 0 \tag{19}$$

The quantity $\sum p_i \dot{q}_i - H(q,p,t)$ can be considered as a function f:

$$f(q, \dot{q}, p, \dot{p}, t) = \sum_i p_i \dot{q}_i - H(q,p,t) \tag{20}$$

for which the $2n$ Euler-Lagrange equations are:

$$\left.\begin{aligned} \frac{d}{dt}\left(\frac{\partial f}{\partial \dot{q}_i}\right) - \frac{\partial f}{\partial q_i} &= 0 \\ \frac{d}{dt}\left(\frac{\partial f}{\partial \dot{p}_i}\right) - \frac{\partial f}{\partial p_i} &= 0 \end{aligned}\right\} \tag{21}$$

The function f contains $\dot{q}_i$ only through the $p_i \dot{q}_i$ term and q_i only in H. So the first equation (21) gives:

$$\dot{p}_j + \frac{\partial H}{\partial q_j} = 0 \tag{22}$$

On the other hand there is no explicit dependence of f on $\dot{p}_i$. So the second equation (21) gives:

$$\dot{q}_j - \frac{\partial H}{\partial p_j} = 0 \tag{23}$$

Eqs (22) and (23) are exactly the Hamilton's equations of motion.

<u>Remark</u>: The variational principle leading to the Euler-Lagrange equations require that the variations of the independent variables vanish at the endpoints. In the phase space that would require $\delta q_i = 0$ and $\delta p_i = 0$ at the endpoints, while the Hamilton's principle requires only $\delta q_i = 0$ at the endpoints.
We note that the variation is required to be zero at the endpoints only in order to get rid of the integrated terms arising from the variations in the time derivatives $\dot{q}, \dot{p}$ of the independent variables. While in f (eq. (20)) that corresponds to the modified Hamilton's principle there is no explicit appearence of $\dot{p}_i$. So the second equation (21) and eq. (23) follow from (19) without assuming $\delta p_i = 0$ at the endpoints.

→ The modified Hamilton's principle with L defined in terms of the Hamiltonian (eq. (4)) leads to the Hamilton's equations under the same variation conditions as those in Hamilton's principle.

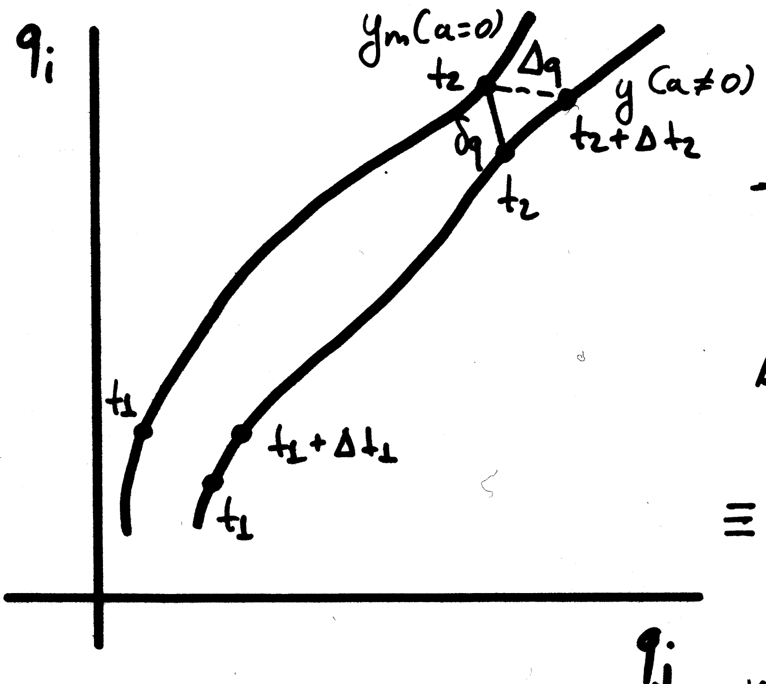
The principle of least action

Δ-variation: The varied path over which an integral is evaluated may end at different times than the correct path, and there may be a variation in the coordinates at the endpoints. Δ-variation occurs in configuration space.

Denoting by y_m the correct path, a family of possible varied paths is defined (as in Hamilton's principle):

$$y_i(t) = y_m(t) + \alpha \cdot n_i(t) \qquad (24)$$

where α is an infinitesimal parameter that goes to zero for the correct path and the functions $n_i(t)$ do not necessarily vanish at the endpoints, either the original or the varied.



The Δ-variation of the action integral is:

$$\Delta \int_{t_1}^{t_2} L\, dt \equiv \qquad (25)$$

$$\equiv \int_{t_1+\Delta t_1}^{t_2+\Delta t_2} L(a)\, dt - \int_{t_1}^{t_2} L(0)\, dt$$

where $L(a)$ means the integral is evaluated along the varied path $y_i(t)$ and $L(0)$ along the actual path y_m.

The variation in eq. (25) composed of 2 parts:

- One arises from the change in the limits of the integral, to first order infinitesimals we have the integrand on the actual path times the difference in the limits in time.
- The second is caused by the change in the integrand on the varied path, but now between the same time limits as the original integral.

So eq. (25) gives:

$$\Delta \int_{t_1}^{t_2} L\,dt = L(t_2)\Delta t_2 - L(t_1)\Delta t_1 + \int_{t_1}^{t_2} \delta L\,dt \qquad (26)$$

Using Hamilton's principle with $\delta q_i \neq 0$ we get

$$\int_{t_1}^{t_2} \delta L\,\delta t = \int_{t_1}^{t_2} \underbrace{\sum_i \left[\frac{\partial L}{\partial q_i} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot q_i}\right)\right]}_{=0} \delta q_i\,dt + \frac{\partial L}{\partial \dot q_i}\,\delta q_i \Big|_1^2 \qquad (27)$$

The Δ-variation becomes:

$$\Delta \int_{t_1}^{t_2} L\,dt = \left(L\Delta t + \sum_i p_i\,\delta q_i \right)\Big|_1^2 \qquad (28)$$

Since

$$\Delta q_i = \delta q_i + \dot q_i\,\Delta t \qquad (29)$$

we get:

$$\Delta \int_{t_1}^{t_2} L\,dt = \left(L\Delta t - \sum_i p_i \dot q_i\,\Delta t + \sum_i p_i\,\Delta q_i \right)\Big|_1^2 \xRightarrow{H = \sum_i p_i \dot q_i - L}$$

$$\Delta \int_{t_1}^{t_2} L\,dt = \left(\sum_i p_i\,\Delta q_i - H\Delta t \right)\Big|_1^2 \qquad (30)$$

To obtain the principle of least action we restrict ourselves to systems such as:

- •) H is not explicit function of time $\Rightarrow$ H is conserved
- •) The variation is such that H is conserved on the varied path as well as on the actual path
- •) We require for the varied paths that

$$\Delta q_i = 0$$

at the endpoints, but not $\Delta t = 0$. This means that the varied path might very well describe the same curve in configuration space as the actual path. The difference will be the speed with which the system point travels on the curve

So eq. (30) gives:

$$\Delta \int_{t_1}^{t_2} L\, dt = -H(\Delta t_2 - \Delta t_1) \qquad (31)$$

Under the above conditions the action integral becomes:

$$\int_{t_1}^{t_2} L\, dt = \int_{t_1}^{t_2} \sum_i p_i \dot{q}_i\, dt - H(t_2 - t_1) \qquad (32)$$

and the Δ-variation gives:

$$\Delta \int_{t_1}^{t_2} L\, dt = \Delta \int_{t_1}^{t_2} \sum_i p_i \dot{q}_i\, dt - H(\Delta t_2 - \Delta t_1) \qquad (33)$$

From eqs. (31) and (33) we get:

$$\Delta \int_{t_1}^{t_2} \sum_i p_i \dot{q}_i \, dt = 0 \tag{34}$$

Principle of Least Action:

The motion of an autonomous Hamiltonian system in the configuration space from time t_1 to time t_2 is such that:

$$\Delta \int_{t_1}^{t_2} \sum_i p_i \dot{q}_i \, dt = 0$$

Remarks

- The above principle is actually a principle of stationary action
- Usually we refer to the integral in Hamilton's principle as the ACTION.
- The variational principles rarely simplify the practical solution of a given mechanical problem. Their value is that they can be used as starting points for new formulation of the theoretical structure of classical mechanics. The most used principle for this purpose is Hamilton's principle.

Canonical Transformations

Consider the situation in which we have a time independent Hamiltonian H where also all coordinates q_i are cyclic:

$$H = H(p_1, p_2, \ldots, p_n) \tag{35}$$

Then we have:

$$-\dot{p}_i = \frac{\partial H}{\partial q_i} = 0 \implies p_i = \alpha_i \text{ (constant)} \tag{36}$$

$$\dot{q}_i = \frac{\partial H}{\partial p_i} = b_i \text{ (constant)} \implies q_i = b_i t + \gamma_i \tag{37}$$

where γ_i are constants determined by the initial conditions

We try to find transformations of the independent coordinates and momenta q_i, p_i to a new set of canonical coordinates Q_i, P_i

$$(q_i, p_i, t) \rightarrow (Q_i, P_i, t) \tag{38}$$

such that the equations of motion in the new set are in the Hamiltonian form:

$$\dot{Q}_i = \frac{\partial K}{\partial P_i} \qquad \dot{P}_i = -\frac{\partial K}{\partial Q_i} \tag{39}$$

for some function $K(Q, P, t)$ which is the transformed Hamiltonian (or Kamiltonian). These transformations are called **CANONICAL**

The idea is to try to find a canonical transformation in order to have many cyclic coordinates Q_i in the new Hamiltonian $K(Q, P, t)$

Since q_i, p_i and Q_i, P_i are canonical coordinates they must satisfy the modified Hamilton's principle:

$$\delta \int_{t_1}^{t_2} \left(\sum_i p_i \dot{q}_i - H(p, q, t)\right) dt = 0 \qquad (40)$$

$$\delta \int_{t_1}^{t_2} \left(\sum_i P_i \dot{Q}_i - K(Q, P, t)\right) dt = 0 \qquad (41)$$

From the above equations we have:

$$\sum_i p_i \dot{q}_i - H = \sum_i P_i \dot{Q}_i - K + \frac{dF}{dt} \qquad (42)$$

where F is any function of the phase space coordinates which is called the generating function of the transformation

We will consider the following forms for the generating function:

$$F_1(q, Q, t) \qquad F_2(q, P, t) \qquad F_3(p, Q, t) \qquad F_4(p, P, t) \qquad (43)$$

•) $F = F_1(q, Q, t)$

Eq. (42) gives:

$$\sum_i p_i \dot{q}_i - H = \sum_i P_i \dot{Q}_i - K + \frac{dF_1}{dt} =$$

$$= \sum_i P_i \dot{Q}_i - K + \frac{\partial F_1}{\partial t} + \sum_i \frac{\partial F_1}{\partial q_i} \dot{q}_i + \sum_i \frac{\partial F_1}{\partial Q_i} \dot{Q}_i \qquad (44)$$

Since q_i, Q_i are independent, the coefficients of $\dot{q}_i, \dot{Q}_i$ must vanish. So we get the equations:

$$\boxed{p_i = \frac{\partial F_1}{\partial q_i} \qquad P_i = -\frac{\partial F_1}{\partial Q_i} \qquad K = H + \frac{\partial F_1}{\partial t}} \qquad (45)$$

•) $F_2 = F_2(q, P, t)$

We consider a generating function that is a function of the old coordinates q and the new momenta P. So we want a relation involving q_i and P_i. This can be done by writing F as:

$$F = F_2(q, P, t) - \sum_i Q_i P_i \tag{46}$$

So we have:

$$\underline{\sum_i p_i \dot{q}_i} - H = \sum_i P_i \dot{Q}_i - K + \frac{d}{dt}\left(F_2 - \sum_i Q_i P_i\right) =$$

$$= \cancel{\sum_i P_i \dot{Q}_i} - K + \underline{\sum_i \frac{\partial F_2}{\partial q_i} \dot{q}_i} + \sum_i \frac{\partial F_2}{\partial P_i} \dot{P}_i - \cancel{\sum_i \dot{Q}_i P_i} - \sum_i Q_i \dot{P}_i + \frac{\partial F_2}{\partial t} \tag{47}$$

Finally we get:

$$\boxed{p_i = \frac{\partial F_2}{\partial q_i} \qquad Q_i = \frac{\partial F_2}{\partial P_i} \qquad K = H + \frac{\partial F_2}{\partial t}} \tag{48}$$

In a similar way we have:

•) $F_3 = F_3(p, Q, t)$

$$F = F_3(p, Q, t) + \sum_i q_i p_i \tag{49}$$

$$\boxed{q_i = -\frac{\partial F_3}{\partial p_i} \qquad P_i = -\frac{\partial F_3}{\partial Q_i} \qquad K = H + \frac{\partial F_3}{\partial t}} \tag{50}$$

•) $F_4 = F_4(p, P, t)$

$$F = F_4(p, P, t) + \sum_i q_i p_i - \sum_i Q_i P_i \tag{51}$$

$$\boxed{q_i = -\frac{\partial F_4}{\partial p_i} \qquad Q_i = \frac{\partial F_4}{\partial P_i} \qquad K = H + \frac{\partial F_4}{\partial t}} \tag{52}$$

Examples of Canonical Transformations

•) Harmonic oscillator in one dimension

$$H = \frac{p^2}{2m} + \frac{m\omega^2}{2} q^2 \tag{53}$$

We note that there is no cyclic coordinate. We consider the generating function:

$$F_1(q, Q, t) = \frac{m}{2}\omega q^2 \cot Q \tag{54}$$

which gives the transformation (eqs. (45)):

$$p = \frac{\partial F_1}{\partial q} = m\omega q \cot Q \qquad P = -\frac{\partial F_1}{\partial Q} = \frac{m\omega q^2}{2\sin^2 Q} \qquad K = H + \frac{\partial F}{\partial t} = H \tag{55}$$

We find q, p with respect to P, Q:

$$q^2 = \frac{2P\sin^2 Q}{m\omega} \qquad p^2 = 2m\omega P\cos^2 Q \tag{56}$$

and the new Hamiltonian becomes:

$$K = H(Q, P) = \omega P \tag{57}$$

where Q is cyclic which means that P=constant. Hamilton's equations give:

$$\dot{P} = -\frac{\partial H}{\partial Q} = 0 \overset{(57)}{\Longrightarrow} P = \frac{h}{\omega}$$

$$\dot{Q} = \frac{\partial H}{\partial P} = \omega \longrightarrow Q = \omega t + \delta \tag{58}$$

where δ is a constant of integration fixed by the initial conditions. From eqs. (56) and (58) we find the old variables as functions of time:

$$q(t) = \sqrt{\frac{2h}{m\omega^2}}\sin(\omega t + \delta) \qquad p(t) = \sqrt{2mh}\cos(\omega t + \delta) \tag{59}$$

where h is the numerical value of H (which is constant)

•) Consider the transformation provided by the generating function:

$$F_1(q, Q) = \sum_i q_i Q_i \qquad (60)$$

The corresponding transformation equations are:

$$p_j = \frac{\partial F_1}{\partial q_j} = Q_j$$
$$P_j = -\frac{\partial F_1}{\partial Q_j} = -q_j \qquad (61)$$

So the new coordinates are the old momenta and the new momenta are the old coordinates

$$Q_j = p_j \qquad P_j = -q_j \qquad (62)$$

The distinction between coordinates and momenta has no direct physical meaning and it is done only for nomenclature reasons.

Poisson Brackets

The Poisson bracket of two functions $f(q,p,t)$ and $g(q,p,t)$ with respect to the canonical variables (q,p) is defined as:

$$[f,g]_{qp} = \sum_i \left(\frac{\partial f}{\partial q_i}\frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q_i} \right) \tag{63}$$

If it is clear which set of canonical variables (q,p) we use we can omit the subscript q,p

From the above definition we get:

$$[f,g] = -[g,f] \tag{64}$$

$$[f,c] = 0 \qquad c = \text{constant} \tag{65}$$

$$\begin{aligned} [f_1+f_2, g] &= [f_1,g] + [f_2,g] \\ [f, g_1+g_2] &= [f,g_1] + [f,g_2] \end{aligned} \tag{66}$$

$$[f_1 \cdot f_2, g] = f_1[f_2,g] + f_2[f_1,g] \tag{67}$$

$$\begin{aligned} \frac{\partial}{\partial t}[f,g] &= \left[\frac{\partial f}{\partial t}, g\right] + \left[f, \frac{\partial g}{\partial t}\right] \\ \frac{d}{dt}[f,g] &= \left[\frac{df}{dt}, g\right] + \left[f, \frac{dg}{dt}\right] \end{aligned} \tag{68}$$

Jacobi's identity: $[f,[g,h]] + [g,[h,f]] + [h,[f,g]] = 0$ (69)

By applying the definition of Poisson brackets we get:

$$[q_i, g] = \frac{\partial g}{\partial q_i} \qquad [p_i, g] = -\frac{\partial g}{\partial p_i} \tag{70}$$

so Hamilton's equations can be written as:

$$\boxed{\dot{q}_i = [q_i, H] \qquad \dot{p}_i = [p_i, H]} \tag{71}$$

For the time derivative of a function we have:

$$\frac{du}{dt} = \sum_i \left(\frac{\partial u}{\partial q_i}\dot{q}_i + \frac{\partial u}{\partial p_i}\dot{p}_i\right) + \frac{\partial u}{\partial t} = \sum_i \left(\frac{\partial u}{\partial q_i}\frac{\partial H}{\partial p_i} - \frac{\partial u}{\partial p_i}\frac{\partial H}{\partial q_i}\right) + \frac{\partial u}{\partial t} \Rightarrow$$

$$\boxed{\frac{du}{dt} = [u, H] + \frac{\partial u}{\partial t}} \tag{72}$$

By putting $u = H$ we get

$$\frac{dH}{dt} = \underbrace{[H, H]}_{=0} + \frac{\partial H}{\partial t} \Rightarrow$$

$$\frac{dH}{dt} = \frac{\partial H}{\partial t}$$

For the canonical coordinates we have:

$$[q_i, q_j] = 0 \qquad [p_i, p_j] = 0 \qquad [q_i, p_j] = \delta_{ij} \tag{73}$$

where $\delta_{ij} = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases}$

For two sets of canonical variables (q,p) and (Q,P) that are connected by a canonical transformation we have:

$$[f,g]_{q,p} = [f,g]_{Q,P} \qquad (74)$$

So from eqs. (73) we get:

$$\left.\begin{aligned} [Q_i, Q_j]_{q,p} &= [Q_i, Q_j]_{Q,P} = 0 \\ [P_i, P_j]_{q,p} &= [P_i, P_j]_{Q,P} = 0 \\ [Q_i, P_j]_{q,p} &= [Q_i, P_j]_{Q,P} = \delta_{ij} \end{aligned}\right\} \qquad (75)$$

The above equations can be used in order to find if a given transformation is canonical or not.

<u>Example</u>

The transformation:

$$Q = \ln p \qquad P = -(1+q+\ln p)p \qquad (76)$$

is canonical, since:

$$[Q,P]_{q,p} = \frac{\partial Q}{\partial q}\frac{\partial P}{\partial p} - \frac{\partial Q}{\partial p}\frac{\partial P}{\partial q} = 0\cdot\frac{\partial P}{\partial p} - \frac{1}{p}(-p) = 1$$

The symplectic formulation of Hamilton's equations

For a systen of n degrees of freedom we construct a column matrix $\underset{\sim}{\eta}$ with n elements such that:

$$\eta_i = q_i \quad , \quad \eta_{i+n} = p_i \qquad i \le n \tag{77}$$

We also construct the matrix $\frac{\partial H}{\partial \eta}$ with elements:

$$\left(\frac{\partial H}{\partial \eta}\right)_i = \frac{\partial H}{\partial q_i} \qquad \left(\frac{\partial H}{\partial \eta}\right)_{i+n} = \frac{\partial H}{\partial p_i} \qquad i \le n \tag{78}$$

Let $\underset{\sim}{J}$ be the $2n \times 2n$ square matrix:

$$\underset{\sim}{J} = \begin{pmatrix} \underset{\sim}{0} & \underset{\sim}{I} \\ -\underset{\sim}{I} & \underset{\sim}{0} \end{pmatrix} \tag{79}$$

where $\underset{\sim}{0}$ is the $n \times n$ matrix all of whose elements are zero, and $\underset{\sim}{I}$ is the $n \times n$ unity matrix. Then Hamilton's equations of motion can be written in a compact form:

$$\boxed{\dot{\underset{\sim}{\eta}} = \underset{\sim}{J} \cdot \frac{\partial H}{\partial \eta}} \Leftarrow \left\{ \begin{array}{l} \text{SYMPLECTIC notation} \\ \text{of Hamilton's equations} \\ \text{of motion} \end{array} \right. \tag{80}$$

$\underset{\sim}{J}$ matrix has the following properties:

$$\underset{\sim}{J}^2 = -\underset{\sim}{I} \qquad \underset{\sim}{J}^T \underset{\sim}{J} = \underset{\sim}{I}$$

$$\underset{\sim}{J}^T = -\underset{\sim}{J} = \underset{\sim}{J}^{-1} \qquad |\underset{\sim}{J}| = +1 \tag{81}$$

where $\underset{\sim}{J}^T$ denotes the transpose of $\underset{\sim}{J}$

The symplectic approach to canonical transformations

Let us consider a canonical transformation in which time does not appear in the equations of transformation

$$Q_i = Q_i(q,p) \qquad (82)$$
$$P_i = P_i(q,p)$$

The new set of coordinates $Q_i P_i$ define a $2n$ element column matrix $\underset{\sim}{\zeta}$, so that eq. (82) can be written as:

$$\underset{\sim}{\zeta} = \underset{\sim}{\zeta}(\underset{\sim}{\eta}) \qquad (83)$$

The time derivative of an element of $\underset{\sim}{\zeta}$ is

$$\dot{\zeta}_i = \sum_j \frac{\partial \zeta_i}{\partial \eta_j} \dot{\eta}_j \qquad i,j = 1, 2, \ldots, 2n \qquad (84)$$

or in matrix notation:

$$\boxed{\dot{\underset{\sim}{\zeta}} = \underset{\sim}{M}\, \dot{\underset{\sim}{\eta}}} \qquad (85)$$

where $\underset{\sim}{M}$ is the Jacobian matrix of the transformation with elements

$$M_{ij} = \frac{\partial \zeta_i}{\partial \eta_j} \qquad (86)$$

Using the equations of motion for η eq. (85) becomes:

$$\dot{\underset{\sim}{\zeta}} = \underset{\sim}{M} \cdot \underset{\sim}{J} \frac{\partial H}{\partial \underset{\sim}{\eta}} \qquad (87)$$

The derivative of H with respect to η_i is:

$$\frac{\partial H}{\partial \eta_i} = \sum_j \frac{\partial H}{\partial \zeta_j} \frac{\partial \zeta_j}{\partial \eta_i} \tag{88}$$

or in matrix notation:

$$\frac{\partial H}{\partial \underset{\sim}{\eta}} = M^T \frac{\partial H}{\partial \underset{\sim}{\zeta}} \tag{89}$$

The combination of eqs. (87) and (89) leads to:

$$\boxed{\dot{\underset{\sim}{\zeta}} = \underset{\sim}{M} \cdot \underset{\sim}{J} \cdot \underset{\sim}{M}^T \frac{\partial H}{\partial \underset{\sim}{\zeta}}} \tag{90}$$

Since the equations of motion for the canonical variables $\underset{\sim}{\zeta}$ is

$$\dot{\underset{\sim}{\zeta}} = \underset{\sim}{J} \frac{\partial H}{\partial \underset{\sim}{\zeta}} \tag{91}$$

we conclude that the transformation $\underset{\sim}{\zeta} = \underset{\sim}{\zeta}(\underset{\sim}{\eta})$ will be canonical if $\underset{\sim}{M}$ satisfies the condition:

$$\boxed{\underset{\sim}{M} \cdot \underset{\sim}{J} \cdot \underset{\sim}{M}^T = \underset{\sim}{J}} \tag{92}$$

Using the properties of $\underset{\sim}{J}$ we see that eq. (92) is equivalent to:

$$\boxed{\underset{\sim}{M}^T \cdot \underset{\sim}{J} \cdot \underset{\sim}{M} = \underset{\sim}{J}} \tag{93}$$

Equations (92) or (93) is spoken of as the <u>symplectic condition</u> for a canonical transformation and the matrix $\underset{\sim}{M}$ satisfying these conditions is said to be a symplectic matrix.